FORMULATION OF THE TWO-DIMENSIONAL TRANSIENT HEAT-CONDUCTION PROBLEM FOR THE BLADED DISK OF A GAS TURBINE

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A system of differential equations is derived for the transient heat conduction associated with a cooled gas turbine disk equipped with shrouded blades on a shank.

The analysis of the solution of the two-dimensional transient heat-conduction problem makes it possible to decide the optimum conditions for the cooling of a turbine rotor. We consider a variable-profile disk having a central orifice and equipped with variable-cross-section shrouded blades. It may be assumed that the disk profile is bounded by parabolas (Fig. 1). Each blade has a shank in the root section. In formulating the problem we take into account the variation of the gas temperature with the height of the cascade as well as the heating of the coolant air. We also assume that the heat-transfer coefficient of the blade web depends on the longitudinal coordinate and varies over the contour of the profile. We take the latter effect into account by asserting that the heat-transfer coefficients involved in the boundary conditions at the fore and aft edges of the blade web are determined by their own intrinsic local heat-transfer conditions, i.e., by critical relations. We assume, in addition, that the variation of the heat-transfer coefficient on the lateral surface of the disk from the center to the periphery is described by an exponential function.

We account for the variation of the gas temperature with the height of the cascade on the basis of the following considerations. The gas in cooled turbines has a maximum temperature roughly in the midsection of the blade web. It may be assumed, therefore, that the blade itself has the maximum temperature in this location. Then the heat flux across the blade midsection is zero, and if we state the boundary condition at this location, it should turn out to be fairly simple. Consequently, we can conditionally partition the entire working portion of the blade into two approximately equal-length parts and formulate the problem for each part separately. In this case the distribution of the gas temperature over the height of the cascade can be described fairly simply and without significant error by means of an exponential function.

In the formulation of the two-dimensional heat-conduction problem the initial differential equation for the blade is derived from the heat balance equation for an elementary part of the web of volume $dV = \delta(x, y)dxdy$. As a result, in place of the function $S(x)$ or $S(y)$ (variation of the blade cross section with the x or y coordinate) we have a single function $\delta(x, y)$ characterizing the blade dimensions in the initial equation.

On the whole, the transient heat-conduction problem for the disk and lower halves of the blades, according to the scheme of Fig. 1, can be represented by a system of two equations:

$$\frac{\partial t_{\mathrm{d}}}{\partial \tau} = a_{\mathrm{d}} \left(\frac{\partial^2 t_{\mathrm{d}}}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial t_{\mathrm{d}}}{\partial r} + \frac{\partial^2 t_{\mathrm{d}}}{\partial z^2} \right), \tag{1}$$

$$r_2 \leqslant r \leqslant r_1, \quad -\delta_2 \leqslant z \leqslant \delta_1 + \delta_2, \quad \tau > 0;$$

$$\frac{\partial t_{\mathrm{b}}}{\partial \tau} = \frac{a_{\mathrm{b}}}{\delta(x, y)} \left\{ \frac{\partial}{\partial x} \left[\delta(x, y) \frac{\partial t_{\mathrm{b}}}{\partial x} \right] + \frac{\partial}{\partial y} \left[\delta(x, y) \frac{\partial t_{\mathrm{b}}}{\partial y} \right] - \frac{2\alpha_3(x)}{\lambda_{\mathrm{b}}} \left[t_{\mathrm{b}} - t^*_{\mathrm{g}}(x) \right] \right\}, \tag{2}$$

$$0 \leqslant x \leqslant l, \quad 0 \leqslant y \leqslant b, \quad \tau > 0.$$

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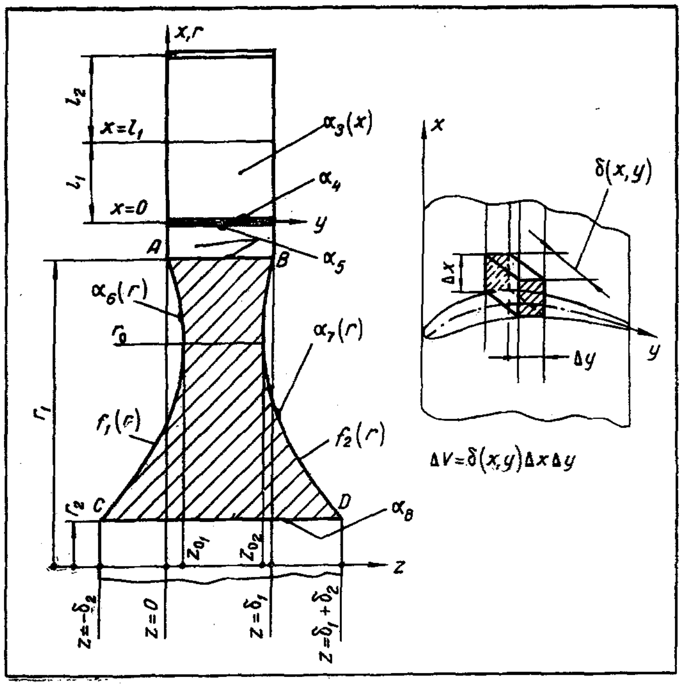

Fig. 1. Diagram relating to the formulation of the problem (α_3, α_4, α_5, α_6, α_7, and α_8 are the heat-transfer coefficients on the corresponding surfaces of the disk and blade).

Here $t_d = f(r, z, \tau)$ and $t_b = f(x, y, \tau)$.

In stating the problem for the upper halves of the blades we need only the second equation. We are concerned here only with the root section of the blade.

For the disk the r coordinate coincides with the radial direction, and z with the axial direction. For the blade the x coordinate coincides in direction with the r coordinate, but the y coordinate is measured along the median of the blade cross section and has its own direction, which does not coincide with the z coordinate. If we take the curvature of the median line into account, then over the entire length y forms a variable angle with the z axis. For the simultaneous solution of Eqs. (1) and (2) it is required to reduce them to coordinate systems the planes of which coincide. This operation can be performed as follows. Figure 2 illustrates the position of the profile orientation of the blade mounted on the disk and the relative position of the y and z axes. The projection of the coordinate system (x, y) onto the plane (r, z) with observance of the y-dependence of the angle φ complicates both the form of the initial blade equation and the solution as a whole. It is therefore convenient to replace the angle φ, which varies along the median line, by a certain angle of inclination φ_0, which the axis of the rectified profile, inscribed within the limits of the disk rim width, forms with the z axis (Fig. 2). The angle φ_0 is not a gas-dynamic characteristic of the cascade. It is a conditional quantity, depending only on the width of the disk rim and the length of the profile along the median line at the base of the blade, i.e., at $x = 0$. This device does not incur a significant error, because in Eq. (2), irrespective of the actual configuration, the y axis is assumed to be straight, and we no longer include the curvature of the median line of the profile in the actual initial equation. Whatever small error does occur is caused only by the difference in the actual distances from the fore edge to the aft edge on the concave and convex sides of the blade. It is important to note that the replacement of the true profile by its conditional counterpart (Fig. 2) in the formulation of the problem is unrelated to the angles of entry β_1 and exit β_2 of the gas flow. It is purely of a formal character. Consequently, the initial equations (1) and (2) and the boundary conditions formulated below are applicable to cascades having blades of any profile. The influence, on the other hand, of the angles β_1 and β_2 and the shape of the blade profile on the heat transfer and, hence, on the temperature field of the blades is taken into account by the critical relations used to calculate the heat-transfer coefficients.

The introduction of the angle φ in the heat-conduction differential equation somewhat alters the form of the latter. We replace y in Eq. (2) by z. According to the scheme of Fig. 2 $z = y \cos \varphi$. It follows, therefore, that if $\varphi = \text{const}$, Eq. (2) must be written in the form

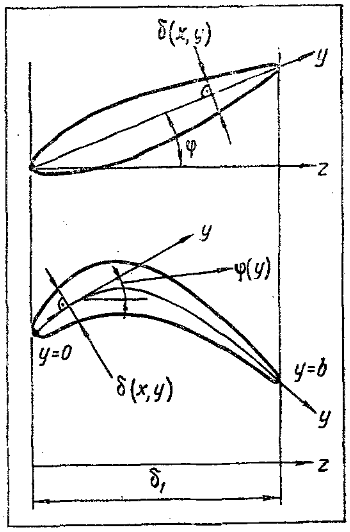

Fig. 2. Illustration of the technique for bringing the plane of the blade (x, y) into coincidence with the coordinate plane of the disk (r, z).

$$\frac{\partial t_{\mathrm{b}}}{\partial \tau}=\frac{a_{\mathrm{b}}}{\delta(x,\ y)}\left\{\frac{\partial}{\partial x}\left[\delta(x,\ y)\frac{\partial t_{\mathrm{b}}}{\partial x}\right]+\cos^2\varphi\frac{\partial}{\partial z}\left[\delta(x,\ y)\frac{\partial t_{\mathrm{b}}}{\partial z}\right]-\frac{2\alpha_3(x)}{\lambda_{\mathrm{b}}}\left[t_{\mathrm{b}}-t_{\mathrm{g}}^*(x)\right]\right\}, \tag{3}$$

$$0\leqslant x\leqslant l,\ 0\leqslant z\leqslant b\cos\varphi,\ \tau>0.$$

Upon rotation of the coordinate plane of the blade into alignment with the plane (r, z) we do not take into account the geometric twist of the blade, i.e., the dependence of φ on x. What is important for the heat-conduction process in the blade and in the disk is the matching of the temperature fields at the junction of the blade with the disk. The variation of the position of the profile of the blade cross section as a function of the x coordinate no longer affects the form of the initial equation or the final result. In other words, the important consideration for the initial equation (2) is its matching with Eq. (1), and it is immaterial how the y axis is rotated for different cross sections of the blade. It is unnecessary, therefore, to take the geometric twist into account in the initial equation.

The series of functions entering into the initial system of differential equations (1) and (2) has the form

$$\alpha_3(x)=A_1-k_1x, \tag{4}$$

$$\delta(x,\ y)=\{(k_2-k_3x)\exp\left[-(k_4+k_5x)\,y\right]+k_6\}\,y, \tag{5}$$

$$t_{\mathrm{g}}^*(x)=A_2\left\{1-\exp\left[-k_z(x+x_0)\right]\right\}. \tag{6}$$

The function $\delta(x, y)$ is determined for the working blades of a particular turbine, but its form can be the same for the blades of other turbines. Only the values of the constant coefficients change.

Upon transition from the y to the z coordinate Eq. (5) acquires the somewhat different form

$$\delta(x,\ y)=\{(k_2-k_3x)\exp\left[-(k_4+k_5x)\,z\cos\varphi\right]+k_6\}\,z\cos\varphi. \tag{7}$$

We now formulate the boundary conditions for the initial equations (1) and (2). The disk is cooled along the lateral surfaces and inside the central orifice, so that the boundary conditions for it are as follows:

at $r = r_2$:

$$\frac{\partial t_{\mathrm{d}}}{\partial r}=\frac{\alpha_8}{\lambda}\left[t_{\mathrm{d}}-t_{\mathrm{a}}(r_2)\right], \tag{8}$$

at $r = r_1$, $x = 0$:

$$\frac{\partial t_{\mathrm{d}}}{\partial r}=C_1t_{\mathrm{d}}+C_2t_{\mathrm{b}}+C_3, \tag{9}$$

at $z = f_1(r)$:

$$\frac{\partial t_{\mathrm{d}}}{\partial n}=\frac{\alpha_6(r)}{\lambda_{\mathrm{d}}}\left[t_{\mathrm{d}}-t_{\mathrm{a1}}(r)\right], \tag{10}$$

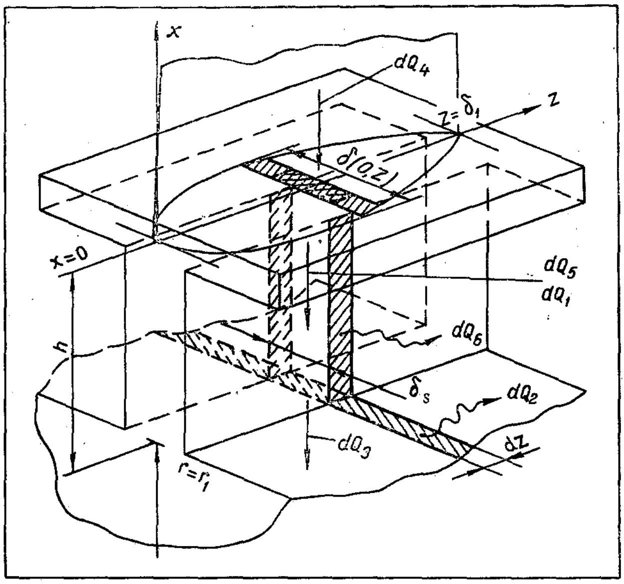

Fig. 3. Diagram relating to the formulation of the heat-balance equations at the junction of a blade with the disk.

at $z = f_2(r)$:

$$-\frac{\partial t_{\mathrm{d}}}{\partial n} = \frac{\alpha_7(r)}{\lambda_{\mathrm{d}}}\left[t_{\mathrm{d}} - t_{\mathrm{a2}}(r)\right]. \tag{11}$$

In Eqs. (8)-(11)

$$\alpha_6(r) = \alpha_z(r) = \alpha_{\mathrm{d}}(r) = \alpha_{\mathrm{d}}(r_2)\, e^{k_{10}(r-r_2)} \tag{12}$$

represents the dependence of the heat-transfer coefficients at the fore and aft lateral surfaces on the radius;

$$f_1(r) = z_{01} - k_{11}(r - r_0)^2, \tag{13}$$

$$f_2(r) = z_{02} + k_{11}(r - r_0)^2 \tag{14}$$

are functions delimiting the disk profile.

In the latter functions

$$r_0 = \left[\delta_2 + k_{11}\left(r_1^2 - r_2^2\right)\right] / \, 2k_{11}(r_1 - r_2), \tag{15}$$

$$z_{01} = k_{11}(r_1 - r_0)^2 \text{ and } z_{02} = \delta_1 - z_{01}. \tag{16}$$

Also

$$t_{\mathrm{a1}}(r) = t_{\mathrm{a1}}(r_2) + k_{12}(r - r_2), \tag{17}$$

$$t_{\mathrm{a2}}(r) = t_{\mathrm{a2}}(r_2) + k_{13}(r - r_2) \tag{18}$$

are functions describing the variation of the air temperature during motion along the lateral surface of the disk from the center toward the periphery. Here

$$t_{\mathrm{a1}}(r_2) = t_{\mathrm{a2}}(r_2) = t_{\mathrm{a}}(r_2). \tag{19}$$

The boundary condition (9) is obtained by solving the heat-balance equation for unit time:

$$dQ_1 = dQ_2 + dQ_3, \tag{20}$$

where

$$dQ_1 = \lambda_{\mathrm{b}} \frac{t_{\mathrm{b}}(0, z, \tau) - t_{\mathrm{d}}(r_1, z, \tau)}{h}\, \delta_s n dz$$

is the heat delivered from the blades to the disk through an elementary cross section of the shank of each blade (Fig. 3);

$$dQ_2=\alpha_5\left[t_d(r_1,\ z,\ \tau)-\bar{t}_a(r_1)\right]b_1ndz$$

is the heat delivered from an elementary outer surface of the rim of dimension b_1dz by the coolant air injected through the duct in the lower part of the blades $(\bar{t}_a(r_1) = 0.5[t_{a1}(r_1) + t_{a2}(r_1)])$;

$$dQ_3=-\lambda_d\frac{\partial t_d(r_1,\ z,\ \tau)}{\partial r}\beta 2\pi r_1dz$$

is the heat transported to the disk from the periphery through the annular surface.

Performing a substitution and transformation, we deduce Eq. (9) from (20), where

$$C_1=-\left(\frac{\lambda_b\delta_s}{h}+\alpha_5b_1\right)\frac{n}{\lambda_d\beta 2\pi r_1};$$
$$C_2=\frac{\lambda_b\delta_s}{h}\cdot\frac{n}{\lambda_d\beta 2\pi r_1};\quad C_3=\frac{\alpha_5b_1n}{\lambda_d\beta 2\pi r_1}t_a(r_1). \tag{21}$$

On the left-hand sides of the boundary conditions (10) and (11) the temperature gradient is expressed as the derivative of t with respect to the normal to the lateral surface of the disk. This is reasonable, since we specify the condition on an arbitrarily oriented boundary of the disk cross section relative to the coordinate system. The transition from $\partial t_d/\partial n$ to $\partial t_d/\partial z$ in the course of solution of the problem can be realized on the basis of the relation

$$\frac{\partial t_d}{\partial n}=\frac{\partial t_d}{\partial z}\cdot\frac{1}{\cos(n,\ z)}. \tag{22}$$

For the lower part of the blade in contact with the disk the boundary conditions are as follows:

at $x = 0$, $r = r_1$:

$$\frac{\partial t_b}{\partial x}=C_4t_d+C_5t_b-C_6; \tag{23}$$

at $x = l$:

$$\frac{\partial t_b}{\partial x}=0; \tag{24}$$

at $y = 0$ $(z = 0)$:

$$\frac{\partial t_b}{\partial z}=\frac{\alpha_3(x,\ 0)}{\lambda_b}\cdot\frac{1}{\cos\varphi}\left[t_b-t_g^*(x)\right]; \tag{25}$$

at $y = b$ $(z = b\cos\varphi)$

$$-\frac{\partial t_b}{\partial z}=\frac{\alpha_3(x,\ b)}{\lambda_b}\cdot\frac{1}{\cos\varphi}\left[t_b-t_g^*(x)\right]. \tag{26}$$

In Eqs. (23)-(26), $\alpha_3(x, 0)$ and $\alpha_3(x, b)$ are the heat-transfer coefficients at the fore and aft edges. By the condition of the problem α_3 does not depend on y, but for the entire surface along the concave and convex sides of the blade it will have one value, while for the fore and aft edges of the blade it will have different values, in accordance with the recommendations of [2].

The function t_g^* has been displayed above [Eq. (6)].

The boundary condition (23) is obtained from the heat-balance equation written for the cross section coinciding with the coordinate $x = 0$ (Fig. 3):

$$dQ_4=dQ_5+dQ_6. \tag{27}$$

Here

$$dQ_4=\lambda_b\frac{\partial t_b}{\partial x}n\delta(0,\ z)\,dz$$

is the heat admitted from an element of the working part of the blade of cross section $\delta(0, z)dz$ to its shank per unit time;

$$dQ_5=\lambda_b \frac{t_b(0, z, \tau)-t_d(r_1, z, \tau)}{h} n\delta_s dz$$

is the heat delivered from the cross section x = 0 to the blade shank through the area $\delta_s dz$;

$$dQ_6=2\alpha_5 \left[t_b(0, z, \tau)-\frac{t_b(0, z, \tau)-t_d(r_1, z, \tau)}{2}-\bar{t}_a(r_1)\right] nhdz$$

is the heat delivered by the coolant air from the lateral surfaces of the shank.

We obtain Eq. (23) by substituting dQ_4, dQ_5, and dQ_6 into it and instituting the appropriate transformations. The constants C_4, C_5, and C_6 are determined by the following expressions in this case:

$$C_4=\left(\frac{\alpha_5 h}{2}-\frac{\lambda_b\delta_s}{h}\right)\frac{1}{\lambda_b\delta(0, z)};$$

$$C_5=\left(\frac{\alpha_5 h}{2}+\frac{\lambda_b\delta_s}{h}\right)\frac{1}{\lambda_b\delta(0, z)}; \quad C_6=\frac{\alpha_5 h}{\lambda_b\delta(0, z)}\bar{t}_a(r_1).$$

We adopt the following as the initial condition for the given system of bodies (disk plus blades):

$$t_d(r, z, 0)=t_b(x, y, 0)=t_0.$$

It is impossible to solve the stated problem in general form. Numerical methods are the most suitable here. In the given case the use of the net-point method enabled us to obtain for the stated problem a numerical solution confirming the validity of all the basic considerations advanced above.

NOTATION

β_1, β_2	are the gas-flow entry and exit angles measured between the direction of motion of the gas and the front of the blade cascade;
a_d, a_b	are the thermal diffusivities of the disk and blade;
τ	is the time;
λ_d, λ_b, c_d, c_b, ρ_d, ρ_b	are the thermal conductivities, specific heats, and densities of the disk and blades;
$t_g^*(x)$	is the stagnation temperature of the gas flow;
$t_a(r)$	is the temperature of the coolant air;
$\delta(x, y)$	is the blade web thickness;
b	is the length of the median line of the profile;
l	is the length of the lower root section of the blade web, equal to half of the total blade length;
δ_s	is the thickness of the blade shank;
b_1	is the distance between the shanks of adjacent blades, measured along the outer cylindrical surface of the disk rim;
n	is the number of blades;
β	is a coefficient accounting for the thermal resistance of the blade root fittings;
h	is the height of the blade shank;
k_1, k_2, . . . , k_7	are constants.

LITERATURE CITED

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